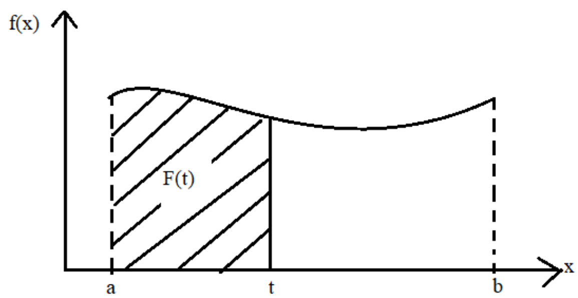
CUMULATIVE DISTRIBUTION FUNCTION, F(x)

If X is a continuous random variable with p.d.f, f(x), cumulative distribution function, F(x) for a value of t in the range of the function is given by;

$$F(t) = P(X \leq t) = \int_{-\infty}^{t} f(x)dx$$

In practice, the lower limit, $-\infty$ is the smallest possible value of x in the range for which x is valid. So if f(x) is valid in the range $a \leq x \leq b, then\ F(t) = \int_a^t f(x)dx$



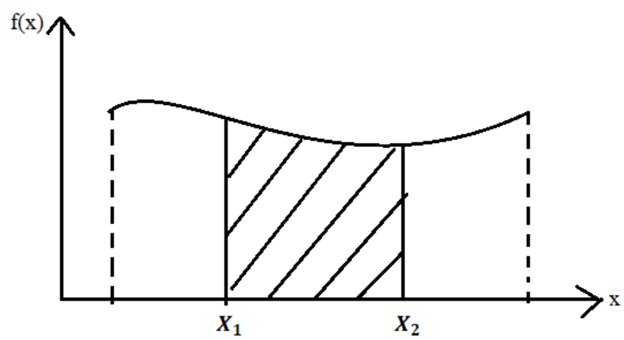
Note: F(t) gives the area under the curve, f(x) up to a particular value t

F(b) = P(($X \leq b$)

= $\int_a^b f(x)dx$

= 1

Finding $P(X_1 \leq X \leq X_2)$ using F(x)



$$P(X_1 \leq X \leq X_2) = F(X_2) - F(X_1)$$

Finding the median, Quartiles and Percentiles

Median is the value 50% of the way through the distribution. If m is the median, then for f(x) defined for $a \leq x \leq b,$

$$\int_a^m f(x)dx = 0.5 \quad \rightarrow F(m) = 0.5$$

The lower quartile, q_1 is the value 25% of the way through the distribution

$$\int_a^{q_1} f(x)dx = 0.25 \quad \rightarrow F(q_1) = 0.25$$

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The lower quartile, q_3 is the value 75% of the way through the distribution

$$\int_a^{q_3} f(x)dx = 0.75 \quad \rightarrow F(q_3) = 0.75$$

Percentiles divide a set of distribution into 100 equal parts

Hence $F(n^{th}\ percentile) = \frac{n}{100}$ e.g $F(10^{th}) = 0.1$, $F(70^{th}) = 0.7$, etc

Examples

1. The continuous random variable X has a p.d.f given by;

$$f(x) = \begin{cases} k(4-x^2), & 0 \le x \le 2 \\ 0 & otherwise \end{cases}$$

Where k is a constant

(a) Show that k = $\frac{3}{16}$

(b) Find the cumulative distribution function, F(x)

(c) Determine P(0.6<X<1.8)

Solution

(a) $\int_0^2 k(4-x^2)dx = 1,$

$$k\left[4x - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[\left(4x2 - \frac{2^3}{3}\right) - 0\right] = 1 \quad \therefore k = \frac{3}{16}$$

(b) $F(t) = \int_0^t \frac{3}{16}(4-x^2)dx = \frac{3}{16}\left[4x - \frac{x^3}{3}\right]_0^t = \frac{3}{16}\left[\left(4t - \frac{t^3}{3}\right) - \left(4x0 - \frac{0^3}{3}\right)\right]$

$$= \frac{3}{16}\left(4t - \frac{t^3}{3}\right)$$

Hence $F(x) = \frac{3}{16}\left(4x - \frac{x^3}{3}\right)$

(c) $P(0.6 < X < 1.8) = F(1.8) - F(0.6) = \frac{3}{16}\left[\left(4x1.8 - \frac{1.8^3}{3}\right) - \left(4x0.6 - \frac{0.6^3}{3}\right)\right]$

= **0.5490**

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2. The probability density function of a continuous random variable x is given by;

$$f(x) = \begin{cases} \frac{2}{13}(x+1), 0 \le x \le 2 \\ \frac{2}{13}(5-x), 2 \le x \le 3 \\ 0 \qquad elsewhere \end{cases}$$

(i) Find the cumulative function and hence, sketch F(x)

(ii) Calculate the median and 20th percentile

(iii) Find P(x<2.5)

Solution

(i) In the range $0 \le x \le 2$

$$\text{F(t)} = \int_0^t \frac{\mathbf{2}}{\mathbf{13}}(\boldsymbol{x}+\mathbf{1})\boldsymbol{dx} = \frac{\mathbf{2}}{\mathbf{13}}\left[\frac{\boldsymbol{x}^{\mathbf{2}}}{\mathbf{2}}+\boldsymbol{x}\right]_{\mathbf{0}}^{\boldsymbol{t}} = \frac{\mathbf{2}}{\mathbf{13}}\left[\left(\frac{t^2}{2}+t\right)-0\right] = \frac{\mathbf{2}}{\mathbf{13}}\left(\frac{t^2}{2}+t\right)$$

This implies; $\boldsymbol{F}(\boldsymbol{x}) = \frac{\mathbf{2}}{\mathbf{13}}\left(\frac{\boldsymbol{x}^{\mathbf{2}}}{\mathbf{2}}+\boldsymbol{x}\right) = \frac{\mathbf{1}}{\mathbf{13}}(\boldsymbol{x}^{\mathbf{2}}+\mathbf{2}\boldsymbol{x})$

$\boldsymbol{F}(\mathbf{2}) = \frac{\mathbf{1}}{\mathbf{13}}(\mathbf{4}+\mathbf{4}) = \frac{\mathbf{8}}{\mathbf{13}}$

In the range $2 \le x \le 3$

$$\text{F(t)} = \text{F(2)} + \int_2^t \frac{\mathbf{2}}{\mathbf{13}}(\mathbf{5}-\boldsymbol{x})\boldsymbol{dx} = \frac{\mathbf{8}}{\mathbf{13}} + \frac{\mathbf{2}}{\mathbf{13}}\left[\mathbf{5}\boldsymbol{x}-\frac{\boldsymbol{x}^{\mathbf{2}}}{\mathbf{2}}\right]_{\mathbf{2}}^{\boldsymbol{t}}$$

$$= \frac{\mathbf{8}}{\mathbf{13}} + \frac{\mathbf{2}}{\mathbf{13}}\left[\left(5t-\frac{\boldsymbol{t}^{\mathbf{2}}}{\mathbf{2}}\right)-\left(5x2-\frac{\mathbf{2}^{\mathbf{2}}}{\mathbf{2}}\right)\right]$$

$$= \frac{\mathbf{8}}{\mathbf{13}} + \frac{\mathbf{2}}{\mathbf{13}}\left(5t-\frac{\boldsymbol{t}^{\mathbf{2}}}{\mathbf{2}}-\mathbf{8}\right) = \frac{\mathbf{1}}{\mathbf{13}}(10t-\boldsymbol{t}^{\mathbf{2}}-\mathbf{8})$$

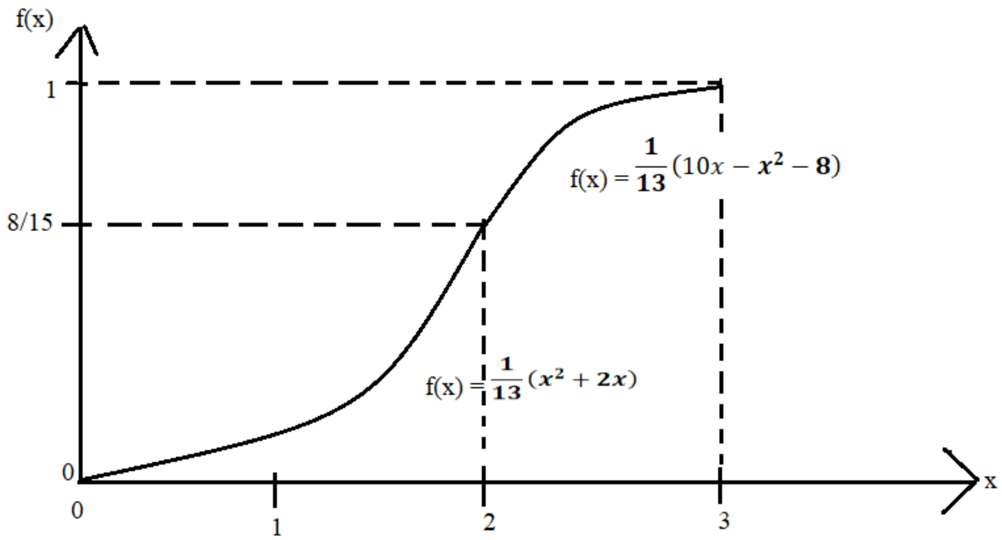
This implies; $F(x) = \frac{\mathbf{1}}{\mathbf{13}}(10x-\boldsymbol{x}^{\mathbf{2}}-\mathbf{8})$

$F(3) = \frac{\mathbf{1}}{\mathbf{13}}(10x3-\mathbf{3}^{\mathbf{2}}-\mathbf{8}) = 1$

Hence;

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{cases} \mathbf{0} & \boldsymbol{x} \le \mathbf{0} \\ \frac{\mathbf{1}}{\mathbf{13}}(\boldsymbol{x}^{\mathbf{2}}+\mathbf{2}\boldsymbol{x}); & \mathbf{0} \le \boldsymbol{x} \le \mathbf{2} \\ \frac{\mathbf{1}}{\mathbf{13}}(10x-\boldsymbol{x}^{\mathbf{2}}-\mathbf{8}); & 2 \le x \le 3 \\ \mathbf{1}; & \boldsymbol{x} \ge \mathbf{1} \end{cases}$$

Sketch of F(x)



(ii) Since F(2) = $\frac{8}{15} > 0.5 \quad then \quad F(m) = 0.5$

$\rightarrow \quad \frac{1}{13}(m^2 + 2m) = 0.5$

$\rightarrow m^2 + 2m - 6.5 = 0$

$\rightarrow m = \frac{-2 \mp \sqrt{2^2 - 4x1x - 6.5}}{2x1} = 1.7386 \; or - 3.7386$

$\boldsymbol{Hence\ median = 1.7386}$

20[th] percentile, $\boldsymbol{F(P_{20}) = 0.2}$

$\frac{1}{13}\left({P_{20}}^2 + 2(P_{20})\right) = 0.2$

${P_{20}}^2 + 2(P_{20}) - 2.6 = 0$

$P_{20} = \frac{-2 \mp \sqrt{2^2 - 4x1x - 2.6}}{2x1} = 0.8974 \; or - 2.8974$

Hence $\boldsymbol{P_{20} = 0.8974}$

(iii) P(x<2.5) = F(2.5) = $\frac{\mathbf{1}}{\mathbf{13}}(10x2.5 - \mathbf{2.5^2} - \mathbf{8}) = 0.8269$

3. The continuous random variable X has cumulative distribution function F(x).

$$F(x) = \begin{cases} 0, & X \le 1 \\ \frac{(x-1)^2}{12} & ;1 \le x \le 3 \\ \frac{14x - x^2 - 25}{24} & ;3 \le x \le 7 \\ 1 & ;x \ge 7 \end{cases}$$

Find the;

(a) median of X

(b) P (x>2.5 /x<4)

(c) interquartile range

Solution

(a) Let m be the median.

Testing for $F(m) \ge 0.5$

$$F(3) = \frac{(3-1)^2}{12} = 0.3333 < 0.5$$

Hence the median is in the range $1 \le x \le 3$

$$F(m) = \frac{14m - m^2 - 25}{24} = 0.5$$

$$14m - m^2 - 25 = 12$$

$$m^2 - 14m + 37 = 0$$

Solving the quadratic equation yields; m = 10.4641 or 3.5359

(b) $\boldsymbol{P\left({}^{x > 2.5}/_{x < 4}\right) = \frac{P(x>2.5\ n\ x<4)}{P(x<4)} = \frac{P(2.5<x<4)}{P(x<4)} = \frac{F(4)-F(2.5)}{F(4)}}$

$$= \frac{\frac{14x4-4^2-25}{24} - \frac{(2.5-1)^2}{12}}{\frac{14x4-4^2-25}{24}} \quad = \mathbf{0.7}$$

(c) Let q_1 and q_3 be the lower and upper quartiles respectively

$$F(q_1) = 0.25$$

$\frac{(q_1-1)^2}{12} = 0.25 \rightarrow (q_1 - 1)^2 = 3 \quad \therefore \boldsymbol{q_1 = 2.7321}$

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$$F(q_3) = 0.75$$

$$\frac{14q_3 - (q_3)^2 - 25}{24} = 0.75 \qquad \rightarrow (q_3)^2 - 14q_3 + 43 = 0$$

$$\rightarrow \; q_3 = 9.4495 \text{ or } 4.5505$$

$$\therefore \boldsymbol{q_3 = 4.5505}$$

Hence Interquartile range = $(\boldsymbol{q_3 - q_1}) = (\mathbf{4.5505 - 2.7321}) = \mathbf{1.8184}$

Trial questions

1. The continuous random variable X has a pdf given by;

$$f(x) = \begin{cases} cx^2; & 0 \le x \le 2 \\ 2c(4-x) & ; 2 \le x \le 4 \\ 0\,; & otherwise \end{cases}$$

where c is a constant

Find the;

(a) value of c
(b) lower quartile
(c) P(1.5<x<2.7)

2. X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} 2kx; 0 \le x \le 1 \\ k(x^2 - 4x + 5); 1 \le x \le 3 \\ 2k(4-x); 3 \le x \le 4 \\ 0; elsewhere. \end{cases}$$

(a) Find the value of the constant k

(b) Find F(x), the cdf of X, hence compute $P(|X-2| < 1.5)$

3. The probability density function of a random variable X is given by;

$$f(x) = \begin{cases} kx^2 & 0.1 \le x \le 0.25 \\ \frac{1}{4}k(0.5-x) & 0.25 \le x \le 0.5 \\ 0 & \text{Otherwise} \end{cases}$$

Find the;

(a) value of k and hence sketch the function F(x)
(b) 30^{th} percentile and semi interquartile range of X

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4. A random variable X has the cumulative distribution function given below.

$$F(x) = \begin{cases} 0 & ;x<0 \\ \frac{x^2}{16} & ;0\leq x \leq 2 \\ ax - b & ;2 \leq x \leq 4 \\ \frac{3}{4}x - \frac{1}{16}x^2 - \frac{5}{4} & ; 4 \leq x \leq 6 \\ 1 & 6<x \end{cases}$$

Find (i) the value of the constants a and b

(ii) $P\ (1\leq x \leq 5/x \geq 2)$

5. The probability density function of a random variable X is given by;

$$f(x) = \begin{cases} kx; & 0 < x < 2 \\ 2k(x-1)^2 & ;2 < x < 5 \\ 0; & elsewhere \end{cases}$$

Find the;

(a) value of constant k
(b) cumulative function, F(x)
(c) $P(|x-3| < 1)$

Finding p.d.f, f(x) from Cumulative function F(x)

Since F(x) can be got by integrating f(x), then f(x) can be obtained by differentiating F(x)

$$\boldsymbol{f(x) = \frac{d}{dx}\left(F(x)\right) = F^1(x)}$$

Examples

1. The continuous random variable X has a cumulative distribution function F(x) given by;

$$F(\boldsymbol{x}) = \begin{cases} \boldsymbol{0} & \boldsymbol{x \leq 0} \\ \boldsymbol{\frac{2x}{3}} & \boldsymbol{0 \leq x \leq 1} \\ \boldsymbol{\frac{x}{3} + k} & \boldsymbol{1 \leq x \leq 2} \\ \boldsymbol{1} & \boldsymbol{x \geq 2} \end{cases}$$

Find the;

(a) value of k
(b) p.d.f, f(x) and sketch it

Solution

(a) At x = 1

$$\frac{\mathbf{2(1)}}{\mathbf{3}} = \frac{\mathbf{1}}{\mathbf{3}} + \mathbf{k} \quad \therefore \mathbf{k} = \frac{\mathbf{1}}{\mathbf{3}}$$

(b) In the range $0 \le x \le 1$

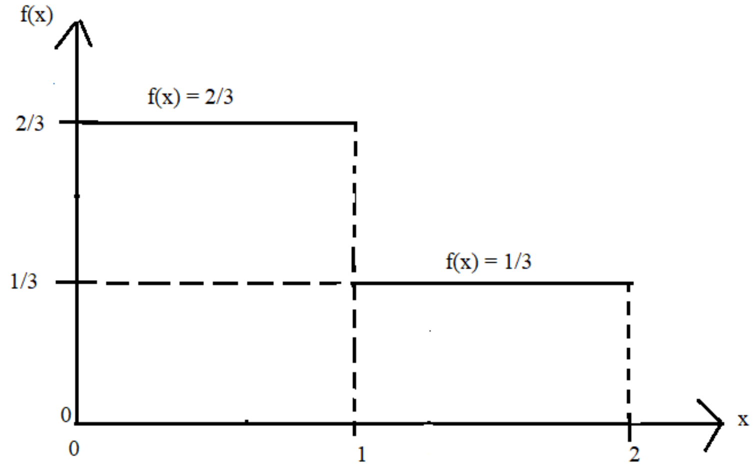
$$f(x) = \frac{d}{dx}\left(\frac{2x}{3}\right) = \frac{2}{3}$$

In the range $1 \le x \le 2$

$$f(x) = \frac{d}{dx}\left(\frac{x}{3} + \frac{1}{3}\right) = \frac{1}{3}$$

Hence $f(x) = \begin{cases} \frac{2}{3}, & 0 \le x \le 1 \\ \frac{1}{3}, & 1 \le x \le 2 \\ 0, & else\ where \end{cases}$

Sketch of f(x)



2. The continuous random variable X has cumulative distribution, F(x) is given by;

$$F(x) = \begin{cases} 0; & x \le 1 \\ \frac{(x-1)^2}{12}; & 1 \le x \le 3 \\ \frac{(14x - x^2 - 25)}{24}; & 3 \le x \le 7 \\ 1; & x \ge 7 \end{cases}$$

Find the;
(a) p.d.f, f(x) and sketch it
(b) E(X)
(c) P(2.8<X<5.2)

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Solution

(a) In the range: $1 \le x \le 3$;

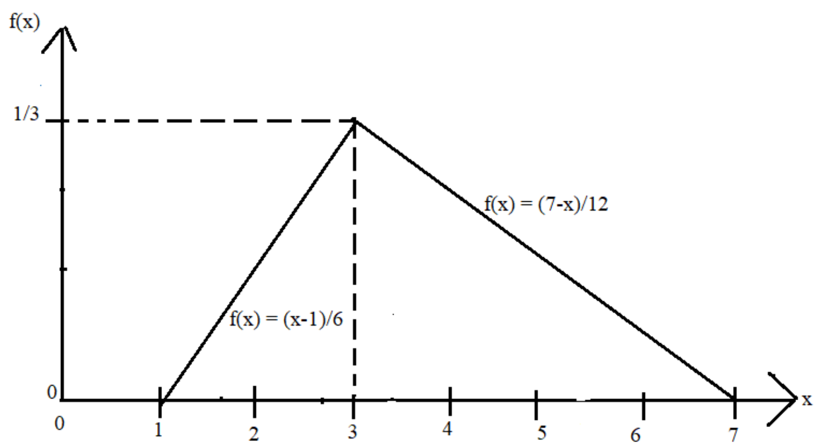
$$f(x) = \frac{d}{dx}\left(\frac{x-1}{12}\right)^2 = \frac{x-1}{6}$$

In the range: $3 \le x \le 7$;

$$f(x) = \frac{d}{dx}\left(\frac{14x-x^2-25}{24}\right) = \frac{7-x}{12}$$

Hence $\boldsymbol{f(x)} = \begin{cases} \frac{\boldsymbol{x-1}}{\boldsymbol{6}}; & \boldsymbol{1 \le x \le 3} \\ \frac{\boldsymbol{7-x}}{\boldsymbol{12}}; & \boldsymbol{3 \le x \le 7} \\ \boldsymbol{0}; & \boldsymbol{otherwise} \end{cases}$

Sketch of f(x)



(b) $$E(x) = \int_1^3 x\left(\frac{\boldsymbol{x-1}}{\boldsymbol{6}}\right)dx + \int_3^7 x\left(\frac{\boldsymbol{7-x}}{\boldsymbol{12}}\right)dx$$

$$= \frac{1}{6}\int_1^3 (x^2 - x)dx + \frac{1}{12}\int_3^7 (7x - x^2)dx$$

$$= \frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3 + \frac{1}{12}\left[\frac{7x^2}{2} - \frac{x^3}{3}\right]_3^7$$

$$= \frac{1}{6}\left[\left(9 - \frac{9}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right] + \frac{1}{12}\left[\left(\frac{343}{2} - \frac{343}{3}\right) - \left(\frac{63}{2} - 9\right)\right]$$

$$= \frac{\boldsymbol{7}}{\boldsymbol{9}} + \frac{\boldsymbol{26}}{\boldsymbol{9}} = \boldsymbol{3\frac{2}{3}\ or\ 3.6667}$$

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(c) $P(2.8 < X < 5.2) = F(5.2) - F(2.8)$

$$= \frac{14(5.2) - (5.2)^2 - 25}{24} - \frac{(2.8-1)^2}{12}$$

$$= \mathbf{0.5950}$$

Trial questions

1. A random variable X has a cumulative distribution function given below.

$$F(x) = \begin{cases} 0, & x \leq \quad 0 \\ ax, & 0 \leq \quad \times \quad \leq 1 \\ \frac{x+b}{3} & 1 \leq \quad \times \quad \leq 2 \\ 1 \quad \times \geq 2 & \end{cases}$$

Find;

(a) the value of a and b

(b) P(X < 1.5/X > 1)

(c) Mean of X

2. The cumulative function of a random variable X is given by;

$$F(x) = \begin{cases} 0; & x < -1 \\ \beta(x+1); & -1 \leq x \leq 0 \\ \beta(2x+1); & 0 \leq x \leq 1 \\ 3\beta; & 1 \leq x \end{cases}$$

Determine the;

(a) Value of β

(b) P.d.f, f(x) of X

(c) Standard deviation, σ of X

(d) $P(|X - \mu| > \frac{1}{3})$

3. A random variable X has a Cumulative distribution function, F(x) given by;

$$F(x) = \begin{cases} 0; & x < 0 \\ \frac{x^2}{2}; & 0 \leq x \leq 1 \\ \left(2x - \frac{x^2}{2} - 1\right); & 1 \leq x \leq 2 \\ 1; & x \geq 2 \end{cases}$$

Find the;

(a) p.d.f, f(x) and sketch it (b) variance of X (c) P(0.5<X<1.3/X>1)