**MODULE 1: ALGEBRA**

QUADRATIC EQUATIONS:

Students are reminded the 3 basic methods of solving quadratics equations and these are:

i) By factorization ii) By completing squares iii) Using the formula

E.g: Solve the equations:

i) $x^2 + 7x + 12 = 0$ the student notes that the factors can easily be derived

$x^2 + 7x + 12 = 0$, $(x + \_\_\ )(x + \_\_) = 0$, $(x+3)(x+4) = 0$, thus $x = -3,\ x = -4$

NOTE: They were cautioned never to miss use the calculators for quadratic equations with no real roots.

E.g. Solve: $x^2 + x + 1 = 0$, this has no real roots. i.e $x = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2}$.

**Miss use of calculators leads them to have such wrong solutions:**

Solve: $2x^2 + 3x + 1 = 0$, $(x + \frac{1}{2})(x+1) = 0$ this is wrong and so is $(x + 0.5)(x+1) = 0$

The correct solution is: $2x^2 + 2x + x + 1 = 0$, to have $2x(x+1) + 1(x+1) = 0$

Thus $(2x+1)(x+1) = 0$ so $x = -\dfrac{1}{2},\ x = -1$

## EQUATIONS LEADING TO QUADRATIC EQUATIONS

1. Solve the equation: $3\frac{3}{5} + \dfrac{4}{x+7} = \dfrac{8}{5-x}$

$\dfrac{18x + 146}{5x + 35} = \dfrac{8}{5-x}$, student to follow up on find the L.C.M

$(18 + 146)(5-x) = 8(5x+35)$, cross multiply and open brackets

$9x^2 + 48x - 225 = 0$, obtain quadratic equation, and use a method to solve

$3x^2 + 16x - 75 = 0$ $\qquad$ $3x^2 - 9x + 25x - 75 = 0$

$(3x+25)(x-3)=0 \qquad x=-\frac{25}{3}, \qquad x=3$

2. Solve the equation: $\frac{y}{3}+\frac{84}{y}=11$, or $y+\frac{252}{y}=33$, or $y^2-33y+252=0$

$y^2-33y+252=0$, to obtain $y^2-21y-12y+252=0$

$(y-12)(y-21)=0$ thus $y=12,\ y=21$

Solve: $y-33\sqrt{y}+252=0$, let $x=\sqrt{y}$

To get $x^2-33x+252=0$

3. Solve the equation: $\frac{x^2}{3}+\frac{84}{x^2}=11$, or $x^2+\frac{252}{x^2}=33$ or $x^4-33x^2+252=0$

Let $y=x^2$

$y^2-33y+252=0$, to obtain $y^2-21y-12y+252=0$

$(y-12)(y-21)=0$ thus $y=12,\ y=21$

Thus, $x^2=12,\ x=\pm 2\sqrt{3}$ or $x^2=21,\ x=\pm\sqrt{21}$

**Common mistake: the students would prefer to write**

$x^4-33x^2+252=0$, **then wrongly factorise** $x^2(x^2-33)=-252$ **then "continue"**

4. Solve the equation: $\frac{x^2+4x}{3}+\frac{84}{x^2+4x}=11$

Let $y=x^2+4x$, $\frac{y}{3}+\frac{84}{y}=11$

$y^2-33y+252=0$, to obtain $y^2-21y-12y+252=0$

$(y-12)(y-21)=0$ thus $y=12,\ y=21$

$x^2+4x-12=0 \qquad x^2+4x-21=0$

$(x+6)(x-2)=0 \qquad (x+7)(x-3)=0$

$x = -6,\ x = 2$ $\qquad\qquad$ $x = -7,\ x = 3$

**Common mistake: complications set in when the equation is not reduced to a quadratic equation.**

TRIAL QUESTIONS

1. Solve the equations:

i) $(x^2 - 2x)^2 + 24 = 11(x^2 - 2x)$ $\qquad$ ii) $(x^2 + 2x) = 34 + \dfrac{35}{(x^2 + 2x)}$

iii) $(3x^2 + 2x)^2 + 8 = 9(3x^2 + 2x)$ $\qquad$ iv) $(x^2 + 2x) = 34 + \dfrac{35}{(x^2 + 2x)}$

IDENTITIES:

Here the students are reminded about the use of the identities

i) $(a + b)^2 = a^2 + 2ab + b^2$ $\qquad$ ii) $(a - b)^2 = a^2 - 2ab + b^2$

TRY

Expand:i) $(2x + 3y)^2$ $\quad$ ii) $\left(2x + \dfrac{3}{x}\right)^2$ $\quad$ iii) $\left(x - \dfrac{2}{x}\right)^2$ $\quad$ iv) $(a + b)^3$

1. Solve the equation: $5x^4 - 21x^3 - 16x^2 - 21x + 5 = 0$ using the substitution $x + \dfrac{1}{x} = y$.

Dividing through by $x^2$: $5x^2 - 21x - 16 - \dfrac{21}{x} + \dfrac{5}{x^2} = 0$

$5\left(x^2 + \dfrac{1}{x^2}\right) - 21\left(x + \dfrac{1}{x}\right) - 16 = 0$ let $y = x + \dfrac{1}{x}$, $y^2 = x^2 + \dfrac{1}{x^2} + 2$

$5(y^2 - 2) - 21y - 16 = 0$ common mistake $y^2 \neq x^2 + \dfrac{1}{x^2}$

$5y^2 - 21y - 26 = 0$, $\quad 5y^2 + 5y - 26y - 26 = 0$

$(5y - 26)(y + 1) = 0$, $\quad y = \dfrac{26}{5}$, $\quad y = -1$

$x+\frac{1}{x}=\frac{26}{5}$, $5x^2-26x+5=0$ $5x^2-25x-x+5=0$

$(5x-1)(x-5)=0$, $x=\frac{1}{5}$, $x=5$

$x+\frac{1}{x}=-1$, $x^2+x+1=0$, $x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2}$ thus has no real roots.

2. Use the substitution $y=x+\frac{1}{x}$ to solve the equation $2x^4-9x^3+14x^2-9x+2=0$

Dividing through by $x^2$, we get

$$2x^2-9x+14-\frac{9}{x}+\frac{2}{x^2}=0$$

But $y^2=x^2+2+\frac{1}{x^2}$

$$\therefore 2\left(x^2+2+\frac{2}{x^2}\right)-9\left(x+\frac{1}{x}\right)+10=0$$

$$2y^2-9y+10=0$$

$$\therefore (y-2)(2y-5)=0$$

If $y=2$

$x+\frac{1}{x}=2$

$(x-1)(x-1)=0$

$\therefore x=1,1$ Note repeated roots

and $y=\frac{5}{2}$,

$x+\frac{1}{x}=\frac{5}{2}$

$(2x-1)(x-2)=0$

$\therefore x=\frac{1}{2},2$

Therefore the roots of the equation are $\therefore x=1,1,\frac{1}{2},2$

NOTE: For quatic equations, some are symmetric while others are not symmetric but can also be reduced to quadratic equations.

TRY:

1. Solve using the substitution $z=x+\frac{2}{x}$ to solve: $x^4-5x^3+10x^2-10x+4=0$

2. Solve: $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

SIMULTANEOUS EQUATIONS

Linear simultaneous equations with 2 unknowns. (Preview of O level mathematics) and the emphasized methods were ELIMINATION and SUBSTITUTION. Matrices will be of little use.

Linear simultaneous with 3 uknowns.(New method is the **row reduction to echelon form**), else we can either use Elimination or substitution method.

1. Solve the equations: $\begin{matrix} 2x - y + 3z = 4 \\ 3x - 2y + 6z = 3 \\ 7x - 4y + 5z = 11 \end{matrix}$ solve to get $x = 5, \; y = 6, z = 0$

2. Solve the simultaneous equations: $\begin{matrix} 4x - y + 8z = 13 \\ x + 3y + 6z = 13 \\ 9x - 10y + 2z = -6 \end{matrix}$

3. Solve the equations simultaneously: $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$, $3x + 4y + 2z = 25$

Let $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4} = t$, so $x = 5t, \; y = 2t - 2, \; z = 4t + 1$

$3(5t) + 4(2t - 2) + 2(4t + 1) = 25$, $31t = 31$ so $t = 1$

$x = 5, \; y = 0, \; z = 5$

4. Solve the equations: $\frac{x-y}{2} = \frac{3y-z}{3} = \frac{2x-z}{5}$ and $3x + 2y - 2z = 10$.

Let $\frac{x-y}{2} = \frac{3y-z}{3} = \frac{2x-z}{5} = k$

$x - y = 2k$, $3y - z = 3k$ and $2x - z = 5k$

$x = 2k + y$ so $2(2k + y) - z = 5k$ thus $2y - z = k$

$y = 2k$, $z = 3k$, $x = 4k$

$12k + 4k - 6k = 10$ to get $k = 1$

$x = 4, \; y = 2, \; z = 3$

TRY:

1. Solve: i) $\frac{12x+7z}{5}=\frac{5y+3z}{3}=\frac{-2x+5z}{2}$, $x+y+z=1$ ii) $\frac{x+2z}{9}=\frac{z+2x}{10}=\frac{x+2y}{14}$, $x+y+z=22$

Note: Further applications involving identities: $\left(\sqrt{x}+\sqrt{y}\right)^2=x+y+2\sqrt{xy}$ and note $\left(\sqrt{x}+\sqrt{y}\right)^2\neq x+y$ which many students use.

1. Solve the equation: $\sqrt{(5x-25)}-\sqrt{(x-1)}=2$

$\sqrt{(5x-25)} = 2+\sqrt{(x-1)}$ *Square both sides*

$(5x-25)=4+4\sqrt{(x-1)}+x-1$

$4x-28 = 4\sqrt{(x-1)}$ *Divide through by 4 and square again*

$x^2-14x+49=x-1$

$x^2-15x+50=0 \quad (x-5)(x-10)=0$

$x=5, x=10$

**<u>Check</u>** for the values $x=5, x=10$ in the equation

If $x=5$, L.H.S = $\sqrt{0}-\sqrt{4}\neq$ R.H.S

2. Solve the equation: $\sqrt{3x+1}+\sqrt{x-1}=\sqrt{7x+1}$

$3x+1+x-1+2\sqrt{(3x+1)(x-1)}=7x+1$

$2\sqrt{(3x+1)(x-1)}=3x+1$

$4(x-1)(3x+1)=(3x+1)(3x+1)$

$(3x+1)[4(x-1)-(3x+1)]=0$

So, $3x+1=0$ $x=-\frac{1}{3}$ and $x-5=0$ $x=5$

**<u>Check:</u>** If $x=5$, L.H.S = $4+2=6=$ R.H.S

If $x = -\frac{1}{3}$, L.H.S $\neq$ R.H.S

$\therefore\ x = 5$, is correct.

The students are reminded to ALWAYS check the solutions at the end with equations involving surds.

TRY:

i) $\sqrt{(3-x)} - \sqrt{(7+x)} = \sqrt{(16+2x)}$ ii) $\sqrt{(x+6)} - \sqrt{(x+3)} = \sqrt{(2x+5)}$

iii) $\sqrt{3(x-2)(x-3)} - \sqrt{(x-2)(x-5)} = (x-2)$

INDICES:

Note the use and application to solving equations.

$2^x \times 2^x = 2^{x+x} = 2^{2x}$, $3^{2x} \times 3 = 3^{2x+1}$, $2 \times 2^x = 2(2^x) = 2^{x+1}$

1. Solve the equations : $\begin{array}{l} 2^x + 4^y = 12 \\ 3(2^x) - 2(2^{2y}) = 16 \end{array}$

$\begin{array}{l} 2^x + 2^{2y} = 12 \\ 3(2^x) - 2(2^{2y}) = 16 \end{array}$ eqn (1)x2 $\begin{array}{l} 2.2^x + 2.2^{2y} = 24 \\ 3(2^x) - 2(2^{2y}) = 16 \end{array}$

Adding: $5.(2)^x = 40$, $(2)^x = 8$, $\therefore\ x = 3$

$8 + 2^{2y} = 12$, $2^{2y} = 4$, $\therefore\ y = 1$

2. Solve the equation $2(3^{2x}) - 5(3^x) + 2 = 0$.

Let $3^x = y$,

$2y^2 - 5y + 2 = 0$

$(2y-1)(y-2) = 0$ $y = \frac{1}{2}$, $y = 2$

$3^x = \frac{1}{2}$, $x = \frac{\log 0.5}{\log 3} = -0.63093$, $3^x = 2$, $x = \frac{\log 2}{\log 3} = 0.63093$

3. Solve for $x$: $3^{2x+1} - 3^{x+1} - 3^x + 1 = 0$

Let $y=3^x$, $3y^2-3y-y+1=0$

$$3y(y-1)-(y-1)=0$$

$$y=\frac{1}{3},\ y=1$$

For $3^x=3^{-1}$, $\Rightarrow x=-1$, $3^x=3^0$, $\Rightarrow x=0$

TRY:

Solve for $x$:

i) $5^{2x+1}+4=21(5^x)$ ii) $2^{2x+8}-32(2^x)+1=0$

iii) $2^{2x+1}-7.2^x+6=0$ iv) $2(5^{2x})-5^{x+1}-3=0$

Solve for $x$ and $y$ given that:

i) $5^x.25^y=1$, $3^{5x}.9=\frac{1}{9}$ ii) $5^{x+2}+7^{y+1}=3468$, $7^y=5^x-76$

iii) $2^x + 3^y = 5$, $2^{x+5}-3^{y+2}=23$

SIMULTANEUOS EQUTIONS II

Linear and non linear with 2 unknowns OR both non linear equations

1. Solve the simultaneous equations: $x-2y=4$, $x^2+3xy+4y^2=2$

$$x=4+2y$$

$$(4+2y)^2+3y(4+2y)+4y^2=2$$

$$16+16y+4y^2+12y+6y^2+4y^2-2=0$$

$$14y^2+28y+14=0$$

$$y^2+2y+1=0$$

$(y+1)^2=0$, $y=-1$, so, $x=2$

ii) $x + y = 10$

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{10}{3}$$

$\frac{\sqrt{x}}{\sqrt{y}} + \frac{\sqrt{y}}{\sqrt{x}} = \frac{10}{3}$, $\frac{x+y}{\sqrt{xy}} = \frac{10}{3}$, thus $\frac{10}{\sqrt{xy}} = \frac{10}{3}$, $\Rightarrow xy = 9$

$\Rightarrow x(10 - x) = 9$, $x^2 - 10x + 9 = 0$

$(x-1)(x-9) = 0$, $x = 1,\ 9$ and $y = 9,\ 1$

iii) Solve the simultaneous equations: $x^2 + xy - 3y^2 = 3$, $x^2 - 2xy + 3y^2 = 9$

$x^2 = 3 - \ xy + 3y^2$ substitute in eqn(ii)

we get $3 - xy + 3y^2 - 2xy + 3y^2 = 9$, $2y^2 - xy = 2$

Thus $x = \frac{2y^2 - 2}{y}$ put $x$ in eqn(i)

$\Rightarrow 3y^4 - 13y^2 + 4 = 0$, $y^2 = \frac{13 \pm \sqrt{169 - 4\times 3\times 4}}{6} = \frac{13 \pm 11}{6} = 4, \frac{1}{3}$

Thus $y = \pm 2$, $x = \pm 3$

Also $y = \pm\frac{1}{\sqrt{3}}$, $x = \pm\frac{4\sqrt{3}}{3}$

iv) Solve the equations:

$$x^2 - 3xy + 2y^2 = 6$$
$$x^2 - xy + y^2 = 21$$

We divide the two equations and we cross multiply:

$\frac{x^2 - 3xy + 2y^2}{x^2 - xy + y^2} = \frac{6}{21}$, i.e $7x^2 - 21xy + 14y^2 = 2x^2 - 2xy + 2y^2$

$5x^2 - 19xy + 12y^2 = 0$, $5x^2 - 15xy - 4xy + 12y^2 = 0$, $(5x - 4y)(x - 3y) = 0$

$y = \frac{x}{3}$ and $y = \frac{5}{4}x$

Case I: $y = \frac{x}{3}$, we get $x^2 - 3x\left(\frac{x}{3}\right) + 2\left(\frac{x}{3}\right)^2 = 6$

Simplify to get $x^2 = 27$, thus $x = \pm 3\sqrt{3}$, and $y = \pm\sqrt{3}$

Case II: $y = \frac{5}{4}x$, we get $x^2 - 3x\left(\frac{5}{4}x\right) + 2\left(\frac{5}{4}x\right)^2 = 6$

Simplify to get $x^2 = 16$, thus $x = \pm 4$, and $y = \pm 5$

**NOTE: There are several alternative ways students can use to get the same answers.**

v) Solve the equations: $x + 2y = 2$
$x^3 + 8y^3 = 56$

$x = 2 - 2y$, so substitute in equation (ii)

$(2 - 2y)^3 + 8y^3 = 56$

$8 - 24y + 24y^2 - 8y^3 + 8y^3 = 56$, simplifying, we have

$y^2 - y - 2 = 0$, to give

$(y+1)(y-2) = 0$

$y = -1, \quad y = 2$ then corresponding values are $x = 4, \quad x = -2$

NOTE: Simultanoues equations will be of later use when finding coordinates of points of intersection between:

Lines, a line and a curve, 2 curves, two circles, three planes, e.t.c,… of which the values obtained will be written as pair of coordinates.

E.g Find the coordinates of the points of intersection between the line $x + 2y = 2$ and the curve $x^3 + 8y^3 = 56$.