P425/1
PURE MATHEMATICS
PAPER 1
JULY 2016
3 HOURS

UGANDA ADVANCED CERTIFICATE OF EDUCATION

INTERNAL MOCK 2016

PURE MATHEMATICS

3 hours

INSTRUCTIONS TO CANDIDATES:

- Attempt **ALL** the **EIGHT** questions in section **A** and any **FIVE** from section **B**.
- All working must be clearly shown.
- Mathematical tables with list of formulae and squared paper are provided.
- Silent, non-programmable calculators should be used.
- State the degree of accuracy at the end of each answer using **CAL** for calculator and **TAB** for tables.
- Clearly indicate the questions you have attempted in a grid on your answer scripts. ***DONOT*** *hand in question paper.*

Qn									
Marks									

SECTION A (40 marks)

1. The sum of the first n terms of an A.P is $n^2 + 5n$. Find the first three terms of the series. (5 marks)

2. Prove that the circles $x^2 + y^2 - 6x - 12y + 40 = 0$ and $x^2 + y^2 - 4y = 16$ are orthogonal. (5 marks)

3. Given that $x = 2$ is a repeated root of the equation $2x^3 + px^2 + qx - 4 = 0$, find the value of p and q. (5 marks)

4. A point $P(1, \ -2, \ 3)$ parallel to the line $\frac{x}{3} = \frac{y+1}{-1} = z + 1$ meets the plane $x + 2y + 2z = 8$ at point Q, find the coordinates of Q. (5 marks)

5. Prove that: $\sin 4\theta = \frac{4\cot\theta(\cot^2\theta - 1)}{(1 + \cot^2\theta)^2}$. (5 marks)

6. Solve the differential equation: $2y(x+1)\frac{dy}{dx} = 4 + y^2$, given that $y = 2$ when $x = 3$. (5 marks)

7. Find the volume generated when the area bounded by the curve $y = 1 + 2x - x^2$ and the line $y = 1$ is rotated through 360^o about the line $y = 1$. (5 marks)

8. Evaluate: $\int_1^2 \frac{x^2 + 2x}{x^3 + 3x^2 - 1}\,dx$. (5 marks)

SECTION B

9. Given that the first three terms in the expansion in ascending powers of x of $(1-8x)^{1/4}$ are the same as the first three terms in the expansion of $\left(\frac{1+ax}{1+bx}\right)$, find the values of a and b. Hence, find an approximation to $(0.6)^{1/4}$ in the form p/q. (12 marks)

10. Express $\dfrac{6-9x}{27x^3+8}$ in partial fractions and hence find $\int \dfrac{6-9x}{27x^3+8}dx$ (12 marks)

11a) Express $\left(-1-\sqrt{3}i\right)^6$ in the form $x+iy$. (6 marks)

b) Given that $z(5+5i)=a(1+3i)+b(2-i)$ where a and b are real numbers and that $\arg z=\dfrac{\pi}{2}$ and $|z|=7$, find the values of a and b. (6 marks)

12a) Given the points $A(-3,\ -3)$, $D(9,\ 5)$ and B such that D divides **AB** externally in the ratio $4:3$, find the coordinates of point B. (4 marks)

b) The vector equation of two lines are $\mathbf{r}=\begin{pmatrix}2\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}2\\2\\t\end{pmatrix}+\mu\begin{pmatrix}1\\2\\1\end{pmatrix}$ where t is a constant. If the two lines intersect find:

(i) t and the position vector of the point of intersection.

(ii) the angle between the two lines. (7 marks)

13a) If ABC is a triangle such that $A+B+C=180^o$, prove that $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$ (6 marks)

b) Express $10\sin x\cos x+12\cos 2x$ in the form $R\sin(2x+\alpha)$, hence or otherwise solve $10\sin x\cos x+12\cos 2x+7=0$ in the range $0^o\le x\le 360^o$. (6 marks)

14. A point P on the curve is given parametrically by $x=3-\cos\theta$ and $y=2+\sec\theta$.

Find the:

(i) equation of the normal to the curve at the point $\theta=\dfrac{\pi}{3}$

(ii) Cartesian equation of the curve. (7 marks)

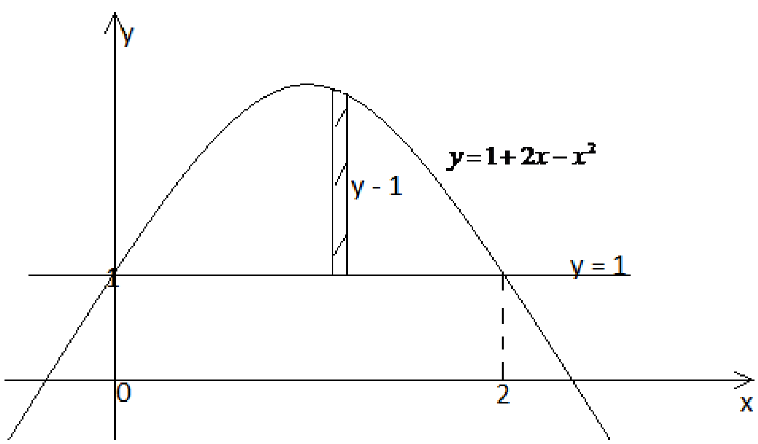
(b) Point $P(ap^2,\ 2ap)$ and $Q(aq^2,\ 2aq)$ lie on the parabola $y^2=4ax$. Find the locus of the midpoint of the chord PQ for which $pq=2a$. (5 marks)

15. Sketch the curve $y = \dfrac{5x^2 + 8x + 4}{x^2 + x}$, by finding the turning points and clearly state the asymptotes. (12 marks)

16. The rate at which a body loses temperature at any instant is proportional to the amount by which the temperature of the body at that instant, exceeds the temperature of its surroundings. A container of hot liquid is placed in a room of temperature $18^o C$ and in 6 minutes the liquid cools from $82^o C$ to $50^o C$. How long does it take for the liquid to cool from $26^o C$ to $20^o C$?

S 6 MARKING GUIDE MOCK TERM 2 2015

1.	$S_n = \frac{n}{2}(2a+(n-1)d)$, $S_2 = 2a+d = 14....(i)$ $S_4 = 4a+6d = 36....(ii)$, $2a+3d=18$ $2d=4$, $d=2$, $a=6$ $6+8+10+...$ ALT The n^{th} term of the progression is $u_n = S_n - S_{n-1}$ Thus $(n^2+5n)-((n-1)^2+5(n-1)) = 2n+4$ So, the terms are $6+8+10+...$	
2.	$x^2+y^2-6x-12y+40=0$, $(x-3)^2+(y-6)^2 = -40+36+9=5$ Thus $C(3, \ 6)$, $r=\sqrt{5}$ $x^2+y^2-4y=16$, $(x-0)^2+(y-2)^2 = 16+4=20$ Thus $C(0, \ 2)$, $r=\sqrt{20}=2\sqrt{5}$ $d=\sqrt{(3-0)^2+(6-2)^2}=5$ $r_1^{\ 2}+r_2^{\ 2}=5+20=25$ Therefore, since $d^2 = r_1^{\ 2}+r_2^{\ 2}$, then the circles are orthogonal.	
3.	$2x^3+px^2+qx-4=0$, $16+4p+2q-4=0$, $2p+q=-6$...(i) $6x^2+2px+q=0$, $24+4p+q=0$, $4p+q=-24$...(ii) $2p=-18$, $p=-9$, $q=12$	
4.	The equation of the line through P is $\mathbf{r}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(3\mathbf{i}-\mathbf{j}+\mathbf{k})$ Thus $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$, so $x=1+3\lambda$, $y=-2-\lambda$, $z=3+\lambda$ $(1+3\lambda)+2(-2-\lambda)+2(3+\lambda)=8$, $\lambda=\frac{5}{3}$ thus $x=6$, $y=-\frac{11}{3}$, $z=\frac{14}{3}$ so $Q\left(6, \ -\frac{11}{3}, \ \frac{14}{3}\right)$	
5.	$\sin 4\theta = \frac{2\sin 2\theta \cos 2\theta}{(\sin^2\theta+\cos^2\theta)^2}$ $= \frac{4\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)}{(\cos^2\theta+\sin^2\theta)^2}$	

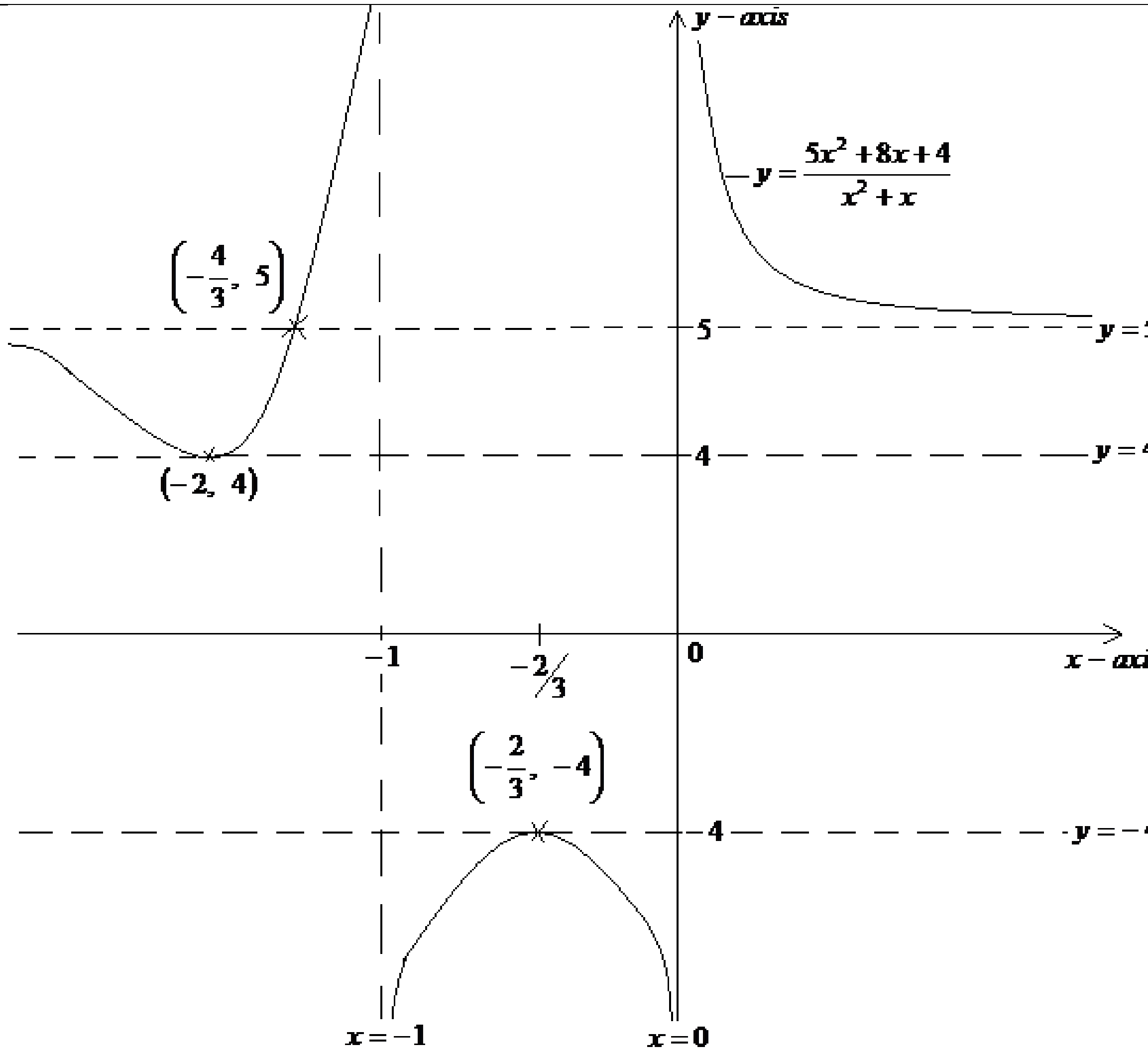
	$$=\frac{\frac{4\sin\theta\cos\theta}{\sin^2\theta}\left(\frac{\cos^2\theta-\sin^2\theta}{\sin^2\theta}\right)}{\frac{(\cos^2\theta+\sin^2\theta)^2}{\sin^4\theta}}$$ $$=\frac{4\cot\theta(\cot^2\theta-1)}{(1+\cot^2\theta)^2}$$	
6	Separating the variables, $$\int\frac{2y}{4+y^2}dy=\int\frac{dx}{x+1}$$ $In(4+y^2)=In(x+1)+Ink$ $y=2$ when $x=3$ $In8=In4+Ink,$ $In2=Ink,$ $k=2$ $In(4+y^2)=In2(x+1)$ thus, $(4+y^2)=2(x+1)$ To get $y=\sqrt{2(x-1)}$	
7.	The point of intersection, $1+2x-x^2=1,$ $x(2-x)=0,$ $x=0,$ $x=2.$  $V=\pi\int_0^2(y-1)^2\ dx,$ $V=\pi\int_0^2(2x-x^2)^2\ dx,$ $V=\pi\int_0^2 4x^2-4x^3+x^4dx,$ $V=\pi\left[\frac{4}{3}x^3-x^4+\frac{x^5}{5}\right]_0^2$ $V=\pi\left(\left(\frac{32}{3}-16+\frac{32}{5}\right)-(0)\right),$ $V=\frac{16}{15}\pi\ cubic\ units$	
8.	Let $u=x^3+3x^2-1,$ $du=3(x^2+2x)dx$ $\int_3^{19}\frac{x^2+2x}{u}\times\frac{du}{3(x^2+2x)}$ $x=2,\ u=19$ $x=1,\ u=3$ $\left[\frac{1}{3}Inu\right]_3^{19}=\frac{1}{3}\left(In\frac{19}{3}\right)$	

9.	$(1-8x)^{1/4} = 1+\frac{1}{4}(-8x)+\frac{1}{2!}.\frac{1}{4}.\frac{-3}{4}(-8x)^2+...$ $= 1-2x-6x^2-...$ $\frac{1+ax}{1+bx}=(1+ax)(1+bx)^{-1} = (1+ax)\{1-bx+b^2x^2+..\}$ $= 1+(a-b)x+(b^2-ab)x^2+...$ Since the first three terms of the expansion are the same. $a-b=-2, b^2-ab=-6$ Hence b = -3 and a = -5 Thus; $(1-8x)^{1/4} \approx \frac{1-5x}{1-3x}$ Substituting $x=0.05$, we have $(1-0.4)^{1/4} \approx \frac{1-0.25}{1-0.15} = \frac{0.75}{0.85}$ hence $(0.6)^{1/4} \approx 15/17$	
10.	$\frac{6-9x}{27x^3+8}=\frac{6-9x}{(3x+2)(9x^2-6x+4)}$ Let $\frac{6-9x}{(3x+2)(9x^2-6x+4)} \equiv \frac{A}{(3x+2)}+\frac{Bx+C}{9x^2-6x+4}$ $6-9x=A(9x^2-6x+4)+(Bx+C)(3x+2)$ $x^2: 9A+3B=0$, $x: -6A+2B+3C=-9$, $x^o: 4A+2C=6$ Solve to get; $A=1$, $B=-3$, $C=1$ $\frac{6-9x}{(3x+2)(9x^2-6x+4)} \equiv \frac{1}{(3x+2)}+\frac{-3x+1}{9x^2-6x+4}$ Thus $\int\frac{6-9x}{27x^3+8}dx=\int\frac{1}{3x+2}dx+\int\frac{-3x+1}{9x^2-6x+4}dx$ $=\frac{1}{3}In(3x+2)+-\frac{1}{6}In(9x^2-6x+4)+c$ Or $In\frac{(3x+2)^{1/3}}{(9x^2-6x+4)^{1/6}}+c$	
11a)	Let $z=-1-\sqrt{3}i$, $\lvert z\rvert=\sqrt{(-1)^2+(-\sqrt{3})^2}=2$ $\arg z=\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)=-\frac{2}{3}\pi$ Thus $-1-\sqrt{3}i=2\left(\cos-\frac{2}{3}\pi+i\sin-\frac{2}{3}\pi\right)$ $(-1-\sqrt{3}i)^6=2^6(\cos-4\pi+i\sin-4\pi)$ $(-1-\sqrt{3}i)^6=64(1+0i)=64+0i$	

b)	$z = 7\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 7i$ $7i(5+5i) = a(1+3i) + b(2-i)$ $-35 + 35i = (a+2b) + i(3a-b)$ $a + 2b = -35\ldots$(i) $3a - b = 35\ldots$(ii) $6a - 2b = 70$ $a = 5$ and $b = -20$	
12a)	$\mathbf{OD} = \dfrac{\mu\mathbf{OA} + \lambda\mathbf{OB}}{\mu + \lambda}$ for $\mathbf{AB}$ ratio is $4:-3$ where $\mu = -3$ and $\lambda = 4$ Let $B(x,\ y)$, $\begin{pmatrix}9\\5\end{pmatrix} = \dfrac{-3\begin{pmatrix}-3\\-3\end{pmatrix} + 4\begin{pmatrix}x\\y\end{pmatrix}}{-3+4}$, $\begin{pmatrix}9\\5\end{pmatrix} = \begin{pmatrix}9\\9\end{pmatrix} + \begin{pmatrix}4x\\4y\end{pmatrix}$ Thus $9 = 9 + 4x$ so $x = 0$ and $5 = 9 + 4y$ so, $y = -1$ The coordinates are $B(0,\ -1)$.	
b)	$\mathbf{r} = \begin{pmatrix}2+\lambda\\1+\lambda\\2\lambda\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}2+\mu\\2+2\mu\\t+\mu\end{pmatrix}$, if they do intersect then $\mathbf{r} = \mathbf{r}$ Thus $\begin{pmatrix}2+\lambda\\1+\lambda\\2\lambda\end{pmatrix} = \begin{pmatrix}2+\mu\\2+2\mu\\t+\mu\end{pmatrix}$ implies that $2+\lambda = 2+\mu$, **so** $\lambda = \mu$ Also $1+\lambda = 2+2\mu$, for $\lambda = \mu$, then $1+\mu = 2+2\mu$ so, $\mu = -1 = \lambda$ Using $2\lambda = t + \mu$, then $-2 = t + -1$, gives $t = -1$ Position vector for point of intersection is given by $\mathbf{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$ or $\mathbf{r} = \mathbf{i} - 2\mathbf{k}$.	
ii)	Let $\mathbf{d}_1 = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{d}_2 = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ be the directional vectors. Thus the angle is given by $\cos\theta = \dfrac{(\mathbf{i}+\mathbf{j}+2\mathbf{k}).(\mathbf{i}+2\mathbf{j}+\mathbf{k})}{\sqrt{1^2+1^2+2^2}.\sqrt{1^2+2^2+1^2}} = \dfrac{5}{6}$ $\cos\theta = \frac{5}{6},\ \theta = 33.56^o$	
13a)	From the L.H.S $\sin^2 A + \sin^2 B + \sin^2 C = \frac{1}{2}(1-\cos 2A) + \frac{1}{2}(1-\cos 2B) + \sin^2 C$ $= 1 - \frac{1}{2}(\cos 2A - \cos 2B) + (1 - \cos^2 C)$ $= 2 - \frac{1}{2}(2\cos(A+B)\cos(A-B)) - \cos^2 C$,	

	$\cos(A+B) = -\cos C$ $= 2 - (-\cos C\cos(A-B)) - \cos^2 C$ $= 2 + \cos C(\cos(A-B) - \cos C)$ $= 2 + \cos C(\cos(A+B) + \cos(A-B))$ $= 2 + 2\cos A\cos B\cos C$	
b)	$10\sin x\cos x + 12\cos 2x = 5\sin 2x + 12\cos 2x$ Let $5\sin 2x + 12\cos 2x \equiv R\sin 2x\cos\alpha + R\cos 2x\sin\alpha$ $\Rightarrow 5 = R\cos\alpha,\ 12 = R\sin\alpha$, thus $\tan\alpha = \frac{12}{5}$ $\therefore \alpha = 67.38^o$ $R = \sqrt{5^2 + 12^2} = 13$ $5\sin 2x + 12\cos 2x = 13\sin(2x + 67.38^o)$ as required. $10\sin x\cos x + 12\cos 2x + 7 = 0$, $13\sin(2x + 67.38^o) = -7$ $2x + 67.38^o = 212.59^o,\ 327.41^o,\ 2x = 145.21^o,\ 260.03^o$ Thus, $x = 72.61^o,\ 130.02^o$	
14.	When $\theta = \frac{\pi}{3}$, $x = 3 - \cos\frac{\pi}{3} = \frac{5}{2}$ and $y = 2 + \sec\frac{\pi}{3} = 4$, thus the point is $P\left(\frac{5}{2},\ 4\right)$. $\frac{dx}{d\theta} = \sin\theta$ and $\frac{dy}{d\theta} = \sec\theta\tan\theta$ thus the gradient function is given by $\frac{dy}{dx} = \sec\theta\tan\theta \times \frac{1}{\sin\theta} = \frac{1}{\cos^2\theta}$ when $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = 4$ Gradient of the normal is $-\frac{1}{4}$, so the equation of the normal is $\frac{y-4}{x-\frac{5}{2}} = -\frac{1}{4}$ To get $8y + 2x = 37$	
	$\cos\theta = 3 - x$, and from $y = 2 + \frac{1}{\cos\theta}$, so $\cos\theta = \frac{1}{y-2}$ Thus $3 - x = \frac{1}{y-2}$ to get $3y + 2x - xy = 7$	
	Let the midpoint be $M(x, y)$, so $x = \frac{ap^2 + aq^2}{2} = \frac{a(p^2 + q^2)}{2}$, $y = \frac{2ap + 2aq}{2} = a(p+q)$ $(p+q) = \frac{y}{a}$, and $2x = a[(p+q)^2 - 2pq]$ but $pq = 2a$	

	Thus $2x = a\left[\left(\frac{y}{a}\right)^2 - 4a\right]$ is the locus of the midpoint.	
15.	$x^2y + xy = 5x^2 + 8x + 4$, $(y-5)x^2 + (y-8)x - 4 = 0$ For no real values, $b^2 - 4ac \le 0$ $(y-8)^2 - 4(y-5)(-4) \le 0$, $y^2 - 16y + 64 + 16y - 80 \le 0$ $y^2 - 16 \le 0$, $(y-4)(y+4) \le 0$ $y < -4$ $-4 < y < 4$ $y > 4$ $y-4$ $-$ $+$ $+$ $y+4$ $-$ $-$ $+$ $sign$ $+$ $-$ $+$ Thus the curve doesnot lie in the region $-4 \le y \le 4$ When $y = -4$, $9x^2 + 12x + 4 = 0$, $(3x+2)^2 = 0$, thus, $x = -\frac{2}{3}$, $\therefore \left(-\frac{2}{3},\ -4\right)\max$ $y = 4$, $x^2 + 4x + 4 = 0$, $(x+2)^2 = 0$, thus, $x = -2$, $\therefore (-2,\ 4)\min$ As $y \to \pm\infty$, $x \to 0$, $x \to -1$, so vertical asymptotes: $x = 0$, & $x = -1$ $y = 5 + \frac{3x+4}{x^2+x}$, As $x \to \pm\infty$, $y \to 5$ horizontal asymptote is $y = 5$, so when $y = 5$, $x = -\frac{4}{3}$, so $\left(-\frac{4}{3},\ 5\right)$ Curve does not cut any of the axes.	



16.

$$\frac{dT}{dt}\ \alpha - (T-18), \quad \frac{dT}{dt} = -K\,(T-18)$$

$$\int \frac{dT}{T-18} = -k\int dt$$

$In(T-18) = -kt + c$ thus $T = 18 + Ae^{-kt}$

When $T = 82^o$, $t = 0$. $A = 64$

$T = 18 + 64e^{-kt}$. When $T = 60^o C$, $t = 6$.

$60 = 18 + 64e^{-6k}$, k=…..

When $T = 26^o C$

$26 = 18 + 64e^{-kt}$, $t = 29.62\text{min}$

When $T = 20^o C$, $t = 49.37\text{min}$

Time to cool is given by $49.37 - 29.62 = 19.75\text{min}$